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# A modified extragradient method for inverse-monotone operators in Banach spaces

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**Abstract** We introduce an iterative procedure for finding a point in the zero set (a solution to $0 \in A(v)$ and $v \in C$) of an inverse-monotone or inverse strongly-monotone operator $A$ on a nonempty closed convex subset $C$ in a uniformly smooth and uniformly convex Banach space. We establish weak convergence results under suitable assumptions.

**Keywords** Extragradient method · Uniformly convex and uniformly smooth Banach space · Weakly sequentially continuous · Inverse-monotone operators · Weak convergence

**Mathematics Subject Classification (2000)** 47H05 · 47J05 · 47J25

## 1 Introduction

Let $X$ be a Banach space with dual space $X^*$ and let $A: X \to X^*$ be a point-to-point operator and $C$ be a nonempty closed convex subset of $X$. The problem of finding $v \in X$ such that

$$Av = 0 \text{ and } v \in C \tag{1.1}$$

is connected with convex minimization problems and variational inequalities (see Lemma 2.2). Iterative methods for finding a point verifying Eq. 1.1 have been extensively studied in [5,7,12,13] and references therein, where the operator $A$ is single-valued or set-valued. The inverse-monotonicity of the nonlinear operators $A$ and strong-nonexpansion of the resolvents $A_\alpha^g$ of $A$, etc., discussed in this paper are generalizations of inverse-strong monotonicity and nonexpansion of operators in Hilbert spaces (see [5,8,11] for details).

For each $\alpha \in (0, \infty)$ we consider the operator $A_\alpha^g: X \to X$ defined by

$$A_\alpha^g x = g^{*\prime}(g'(x) - \alpha Ax) \tag{1.2}$$

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(see [5]), where $g$ is a lower semicontinuous and proper convex function, $g^{*\prime}$ is the derivative of Fenchel conjugate of $g$, and $g$ and $g^*$ are Gâteaux differentiable on $X$ and $X^*$, respectively. Butnariu and Resmerita [5] considered the following iterative process:

$$\text{Choose } x_0 \in C \text{ and define } x_{n+1} = \Pi_C^g \circ A_\alpha^g x_n, \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $\Pi_C^g$ denotes the Bregman projection of $x$ on $C$ with respect to $g$, and "$\circ$" means the composition operation of the operators $\Pi_C^g$ and $A_\alpha^g$. When $A$ is inverse-monotone and weakly sequentially continuous, or it is inverse strongly-monotone, it is shown in [5] that the weak accumulation points of the sequence generated by (1.3) are solutions of Eq. 1.1.

On the other hand, Kamimura, Kohsaka and Takahashi [9] studied the following algorithm in a smooth and uniformly convex Banach space for finding a point in the zero set of a maximal monotone operator $T$; namely,

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(J_{r_n}x_n)), \quad n = 1, 2, \ldots, \tag{1.4}$$

where $J_r = (J + rT)^{-1}J$, $J$ is the duality mapping of $X$ and the sequences $\{\alpha_n\}$, $\{r_n\}$ of real numbers are chosen appropriately.

In [11], when $X$ is a uniformly smooth and uniformly convex Banach space, the authors studied the problem finding $v \in X$ satisfying $v \in T^{-1}0 \cap A^{-1}0 \cap C$, where $A$ is an inverse-monotone operator and $T$ is a maximal monotone operator. To this end, they proposed a hybrid method of extragradient method and proximal point algorithm as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \Pi_C J^{-1}(J(x_n) - \alpha A x_n), \\ z_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}y_n)), \\ x_{n+1} = \Pi_C z_n, \quad n = 1, 2, \ldots, \end{cases} \tag{1.5}$$

where $\Pi_C$ is shortening of $\Pi_C^g$ in the case $g = \frac{1}{2}\|\cdot\|^2$. When $\beta_n = r_n \equiv 0$, the iterative (1.5) reduces to (1.3) with $g = \frac{1}{2}\|\cdot\|^2$. In this paper, we consider only the generalization of the iterative process (1.2) for solving the equation $v \in A^{-1}0 \cap C$.

Motivated by (1.3) and (1.4), we propose an iterative scheme for finding a zero of an inverse-monotone operator $A\colon X \to X^*$ relative to $g = \frac{1}{2}\|\cdot\|^2$ on a closed convex subset $C$ of $X$ in the case when $X$ is a uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0, x_1 \in C, \\ z_n = J^{-1}((1-\beta_n)J(A_\alpha x_n) + \beta_n J(A_\alpha x_{n-1})), \\ x_{n+1} = \Pi_C z_n, \quad n = 1, 2, \ldots, \end{cases} \tag{1.6}$$

where $x_0, x_1 \in C$, $\beta_n \in [0, 1)$, and $\Pi_C$ and $A_\alpha$ are shortening of $\Pi_C^g$ and $A_\alpha^g$ whenever $g = \frac{1}{2}\|\cdot\|^2$. We prove that the sequence generated by (1.6) converges weakly to a solution of Eq. 1.1 under suitable assumptions.

When $\beta_n \equiv 0$, the algorithm (1.6) reduces to (1.3).

## 2 Preliminaries

Suppose that $X$ is a reflexive Banach space with dual space $X^*$, and the norms on $X$ and $X^*$ are denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. As usual, we denote the duality pairing of $X^*$ and $X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$, where $x^* \in X^*$ and $x \in X$, and $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence of a sequence in $X$, respectively. Also, we use $w-\lim_{n\to\infty} x_n$ to denote the weak-limit of a sequence $\{x_n\}$ of $X$. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all nonnegative integers, respectively.

Let $S=\{x\in X \mid \|x\|=1\}$ denote the unit sphere of a Banach space $X$. A Banach space X is said to be uniformly convex if, for any $\varepsilon\in(0,2]$ there exists $\delta>0$ such that, for any $x,y\in S$,

$$|x-y\|\geq\varepsilon \text{ implies } \|\frac{x+y}{2}\|\leq 1-\delta.$$

A Banach space $X$ is said to be strictly convex if, for any $x, y\in S$ and $x\neq y$, $\|\frac{x+y}{2}\|<1$. The norm on $X$ is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for each $x,y\in S$ and in this case $X$ is said to be smooth. $X$ is said to have a uniformly Fréchet differentiable norm if the limit (2.1) is attained uniformly for $x,y\in S$ and in this case $X$ is said to be uniformly smooth.

The related properties of the strict (uniform) convexity, (uniform) smoothness of Banach spaces can be found in [2]. For instance, $X$ is uniformly convex if and only if $X^*$ is uniformly smooth, which implies that $X$ is reflexive; if $X$ is a reflexive Banach space, then $X$ is strictly convex if and only if $X^*$ is smooth and $X$ is smooth if and only if $X^*$ is strictly convex.

The duality mapping $J\colon X\rightrightarrows X^*$ is defined by

$$J(x)=\{x^*\in X^* \mid \langle x^*,x\rangle=\|x^*\|_*^2=\|x\|^2\},\quad \forall\, x\in X;$$

the duality mapping $J^*\colon X^*\rightrightarrows X$ is defined by

$$J^*(x^*)=\{x\in X \mid \langle x,x^*\rangle=\|x\|^2=\|x^*\|_*^2\},\quad \forall\, x^*\in X^*.$$

The following results concerning the duality mapping are well known (see [2,15]): $X$ is reflexive, strictly convex and smooth if and only if $J$ is single-valued and bijective. In this case $J^{-1}=J^*$. If $X$ is uniformly convex, then $J^*$ is uniformly norm to norm continuous on each bounded subset of $X^*$ and $X$ is strictly convex.

A duality mapping $J$ of a smooth Banach space is said to be weakly sequentially continuous if $x_n\rightharpoonup x$ implies that $\{Jx_n\}$ converges weakly* to $Jx$.

**Lemma 2.1** ([16]) *Let $X$ be a uniformly convex Banach space. Then for each $r>0$, there exists a strictly increasing, continuous and convex function $k\colon[0,\infty)\to[0,\infty)$ such that $k(0)=0$ and*

$$\|\lambda x+(1-\lambda)y\|^2\leq\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)k(\|x-y\|)$$

*for all $x,y\in\{z\in X \mid \|z\|\leq r\}$ and $\lambda\in[0,1]$.*

A function $g\colon X\to(-\infty,+\infty]$ is said to be proper if the set $\{x\in X \mid g(x)\in\mathbb{R}\}$ is nonempty. A proper function $g$ is said to be convex if

$$g(\lambda x+(1-\lambda)y)\leq\lambda g(x)+(1-\lambda)g(y) \tag{2.2}$$

for all $x,y\in X$ and $\lambda\in(0,1)$. $g$ is strictly convex, if the inequality (2.2) is strict. Additionally $g$ is said to be lower semicontinuous if the set $\{x\in X \mid g(x)\leq\lambda\}$ is closed in $X$ for all $\lambda\in\mathbb{R}$.

Let $g$ be a lower semicontinuous, proper and strictly convex function. Assume that $g$ is Gâteaux differentiable on $X$, and so its Fenchel conjugate $g^*$ is also Gâteaux differentiable on $X^*$. The Bregman distance with respect to $g$ is defined as $D_g\colon X \times X \to \mathbb{R}$,

$$D_g(x, y) = g(x) - g(y) - \langle g'(y), x - y\rangle. \tag{2.3}$$

As $g$ is strictly convex and Gâteaux differentiable, the function $D_g(\cdot, y)$ for any $y \in X$ is nonnegative, strictly convex and $D_g(x, y) = 0$ if and only if $x = y$.

Given a nonempty closed convex subset $C \subset X$ and any $x \in X$, the Bregman projection of $x$ on $C$ with respect to $g$, which is denoted by $\Pi_C^g x$, is defined as the solution of the convex optimization problem $\min_{y\in C} D_g(y, x)$, i.e.,

$$\Pi_C^g x\colon = \arg\min_{y\in C} D_g(y, x). \tag{2.4}$$

The modulus of total convexity of $g$ is the function $\nu_g\colon X \times \mathbb{R}_+ \to \mathbb{R}$, defined as

$$\nu_g(x, t) = \inf\{D_g(y, x)\colon y \in X, \|y - x\| = t\}. \tag{2.5}$$

A function $g$ is called totally convex at $x$ if $\nu_g(x, t) > 0$ whenever $t > 0$. A function $g$ is called totally convex if $\nu_g(x, t) > 0$ whenever $x \in X$ and $t > 0$. A function $g$ is called totally convex on bounded sets if $\inf_{x\in B} \nu_g(x, t) > 0$ for each bounded nonempty subset $B$ of $X$. The total convexity on bounded sets of $g$ is also termed the uniform total convexity of $g$ (see [6]). If $g$ is totally convex, then $\nu_g(x, st) \geq s\nu_g(x, t)$ for all $s \geq 1, t \geq 0$ and $x \in X$.

The function $g$ is called sequentially consistent [3] if for any two sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n\}$ is bounded and $\lim_{n\to\infty} D_g(y_n, x_n) = 0$ we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Also, $g$ is sequentially consistent if and only if it is totally convex on bounded sets (see Proposition 4.2 in [4]).

Applying the Bregman projection with respect to $g$, where, except the above assumptions, $g$ is also totally convex or coercive in the sense that $\lim_{\|y\|\to\infty} \frac{g(y)}{\|y\|} = \infty$, we know that the operator $\Pi_C^g\colon X \to C \subset X$ is well defined and it holds (see Corollary 4.5 in [5]) that

$$D_g(y, \Pi_C^g x) + D_g(\Pi_C^g x, x) \leq D_g(y, x),\ \forall\, y \in C. \tag{2.6}$$

Let $A\colon X \to X^*$ be an operator and $A_\alpha^g$ be defined as above. When $g = \frac{1}{2}\|\cdot\|^2$, we denote $A_\alpha^g x = A_\alpha x$. Due to $g^{*\prime} = (g')^{-1}$, we know that $Ax = 0$ if and only if $x$ is a fixed point of $A_\alpha^g$.

We say that the operator $A$ is inverse-monotone (see [5]) on $C$ relative to $g$ if there exists a real number $\alpha > 0$ and a vector $z \in C$ such that

$$\langle Ay, A_\alpha^g y - z\rangle \geq 0, \quad \forall y \in C. \tag{2.7}$$

In this case, the vector $z$ involved in (2.7) is called monotonicity pole of $A$. Denote by $A_0$ the set of all such vectors. Clearly, $A_0$ is a closed convex subset of $C$.

If $A$ is inverse-monotone on $C$ relative to $g$, by Theorem 5.5 in [5], it holds true that $A^{-1}0 \cap C \neq \emptyset$. That is the set of solutions of Eq. 1.1 is nonempty.

Let $T\colon X \rightrightarrows X^*$ be a set-valued maximal monotone operator and $C$ be a nonempty closed convex subset of $X$. Usually, We view $N_C x$ as the normal cone for $C$ at a point $x \in C$, i.e.,

$$N_C x = \{x^* \in X^* \mid \langle y - x, x^*\rangle \leq 0 \text{ for all } y \in C\}.$$

Let

$$Sv = \begin{cases} Tv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.8}$$

When $X$ is a reflexive Banach space and $int(Dom(T)) \cap C \neq \emptyset$, it is known that $S$ is maximal monotone. Additionally, if $X$ is strictly convex and smooth as well, we denote the resolvent of $S$ by $J_r = (J + rS)^{-1}J$ for all $r > 0$ and the Yosida approximation of $S$ by $\mathcal{B}_r = \frac{J - JJ_r}{r}$.

**Lemma 2.2** *Suppose $X$ is a reflexive, strictly convex and smooth Banach space. Let $T : X \rightrightarrows X^*$ be a maximal monotone operator and $C$ be a nonempty closed convex subset of $X$. If $S$ defined as (2.8) is maximal monotone and $S^{-1}0 \neq \emptyset$, then $S^{-1}0 = \mathcal{B}_r^{-1}0 \cap C$ for all $r > 0$, and $\mathcal{B}_r$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $r$ and $S^{-1}0 = \mathcal{B}_{r0}$.*

*Proof* Since $v \in S^{-1}0$ if and only if $v = J_r v$ for any $r > 0$, which is equivalent to $\mathcal{B}_r v = \frac{Jv - JJ_r v}{r} = 0$ for any $r > 0$, i.e., $v \in \mathcal{B}_r^{-1}0$ for $r > 0$. Also, $v \in S^{-1}0$ means $v \in C$. Consequently, $v \in S^{-1}0$ if and only if $v \in \mathcal{B}_r^{-1}0 \cap C$ for all $r > 0$.

Let $\mathcal{B}_{rr}x = J^{-1}(Jx - r\mathcal{B}_r x)$. While $g = \frac{1}{2}\|\cdot\|^2$, $g' = J$. By (1.2), this is the case that $\mathcal{B}_{rr}x = \mathcal{B}_{rr}^{g}x$. Since $S$ is maximal monotone and $\mathcal{B}_r x \in S(J_r x)$ for all $x \in X$ and $r > 0$, we have that

$$\begin{aligned} \langle v - \mathcal{B}_{rr}x, \mathcal{B}_r x\rangle &= \langle v - J^{-1}(Jx - r\mathcal{B}_r x), \mathcal{B}_r x\rangle \\ &= \langle v - J_r x, \mathcal{B}_r x\rangle \leq 0 \end{aligned} \tag{2.9}$$

for all $v \in S^{-1}0$, $x \in X$ and $r > 0$, which implies by (2.7) that, for all $r > 0$, $\mathcal{B}_r$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $r$ and $S^{-1}0 \subseteq \mathcal{B}_{r0}$. Since $\mathcal{B}_{r0} \subseteq \mathcal{B}_r^{-1}0 \cap C = S^{-1}0$, we get $\mathcal{B}_{r0} = S^{-1}0$. □

Note that in this case we also have $S^{-1}0 = \mathcal{B}_r^{-1}0$ for all $r > 0$. Consequently, the Yosida approximation $\mathcal{B}_r$ of the maximal monotone operator $S$ is inverse-monotone, moreover, the monotonicity pole $\mathcal{B}_{r0}$ relative to constant $r$ of $\mathcal{B}_r$ is just $S^{-1}0$.

Recall [3] that an operator $B : X \to X$ is called totally nonexpansive with respect to the function $g$ on the set $C$ if there exists a vector $z \in C$ such that

$$D_g(z, Bx) + D_g(Bx, x) \leq D_g(z, x), \forall\, x \in C. \tag{2.10}$$

A vector $z$ for which the condition (2.10) is satisfied is called nonexpansivity pole of $B$ with respect to $g$.

**Lemma 2.3** ([5]) *The operator $A$ is inverse-monotone relative to $g$ with constant $\alpha$ if and only if the operator $A_\alpha^g$ is totally nonexpansive with respect to $g$, that is, for some $z \in C$ the following inequality holds:*

$$D_g(z, A_\alpha^g x) + D_g(A_\alpha^g x, x) \leq D_g(z, x), \forall\, x \in C. \tag{2.11}$$

*In this case, $z \in C$ is a monotonicity pole of $A$ if and only if it is a nonexpansivity pole of $A_\alpha^g$.*

Recall that the operator $A$ is nonexpansive on $C$ with respect to $g$ if

$$D_g(Ax, Ay) \leq D_g(x, y), \quad \forall x, y \in C. \tag{2.12}$$

We say that the operator $A$ is inverse strongly-monotone (see [5]) on $C$ relative to $g$ if $A$ is inverse-monotone on $C$ relative to $g$ with constant $\alpha > 0$ and $A_\alpha^g$ is nonexpansive on $C$ with respect to $g$, i.e.,

$$D_g(A_\alpha^g x, A_\alpha^g y) \leq D_g(x, y), \quad \forall x, y \in C. \tag{2.13}$$

Let $A\colon X \to X^*$ be an operator and $C$ be a nonempty closed convex subset of $X$. The variational inequality problem is to find $x \in C$ such that

$$\langle Ax, v - x\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

**Lemma 2.4** ([11]) *Let $A\colon X \to X^*$ be an operator and $C$ be a nonempty closed convex subset of $X$. If $A^g_\alpha$ is nonexpansive on $C$ with respect to $g$ for some $\alpha > 0$ and $A^{-1}0 \cap C \neq \emptyset$, then $x \in VI(C, A)$ if and only if $x \in A^{-1}0 \cap C$.*

**Lemma 2.5** ([11]) *Let $A\colon X \to X^*$ be an operator and $C$ be a nonempty closed convex subset of $X$. If $A$ is inverse-monotone, then $A^{-1}0 \cap C \neq \emptyset$ and $x \in VI(C, A)$ if and only if $x \in A^{-1}0 \cap C$.*

We say that the operator $A^g_\alpha$ is strongly nonexpansive on $C$ relative to $g$ (where $\alpha > 0$) if there exists $\lambda > \alpha$ such that

$$D_g(A^g_\alpha x, A^g_\alpha y) \leq D_g(x, y) + \alpha(\alpha - \lambda) D_g(g^{*\prime}(Ax), g^{*\prime}(Ay)), \quad \forall x, y \in C \tag{2.14}$$

(see [11]). Clearly, if an operator $A^g_\alpha$ is strongly nonexpansive on $C$ relative to $g$ with constant $\lambda$, then it is nonexpansive on $C$ relative to $g$ with constant $\alpha > 0$.

Recall that $g$ is quadratically homogeneous on $X$, if $g(\alpha x) = \alpha^2 g(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$. It is easy to verify that if $A = g'$, where $g$ is a quadratically homogeneous, lower semicontinuous and proper convex function on $X$, then $A^g_\alpha$ is strongly nonexpansive on $X$ relative to $g$ (see [11]).

We say that the operator $A$ is strongly inverse-monotone on $C$ relative to $g$ if there exists some real number $\alpha > 0$ such that

$$\langle Ax - Ay, A^g_\alpha x - A^g_\alpha y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.15}$$

When $A = g'$ and $0 < \alpha \leq 1$, $A$ is strongly inverse-monotone on $C$ relative to $g$ (see [11]).

**Lemma 2.6** ([11]) *Let $\emptyset \neq A^{-1}0 \cap C$. If an operator $A$ is strongly inverse-monotone on $C$ relative to $g$, then it is inverse-monotone on $C$ relative to $g$ with $A_0 = A^{-1}0 \cap C$.*

The following Lemma is important in our paper.

**Lemma 2.7** ([5]) *Suppose that the function $g : X \to R$ is totally convex on bounded sets and has bounded Gâteaux derivative $g'$ on bounded sets. If $T : C \to X$ is an operator such that*

$$D_g(Ty, Tx) \leq D_g(y, x), \quad \forall x, y \in C,$$

*then for any weakly convergent sequence $\{x_n\} \subseteq C$ which has $\lim\limits_{n\to\infty} D_g(Tx_n, x_n) = 0$, the vector $x = w - \lim\limits_{n\to\infty} x_n$ is a fixed point of $T$.*

The following conclusions are key to our convergence analysis.

**Proposition 2.1** *Let $X$ be a reflexive Banach space and suppose that the function $g : X \to \mathbb{R}$ is lower semicontinuous and totally convex on bounded sets, and $g'$ is uniformly norm to norm continuous on bounded sets. Let $\{x_n\}$ be bounded such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and $C$ be a nonempty closed convex subset of $X$. Suppose that $\{D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)\}$ is* *nonincreasing for any* $v \in C$*, where* $\{\gamma_n\}$ *is a nonnegative bounded sequence. Then* $\{\Pi_C^g x_n\}$ *converges strongly to* $v_0 \in C$*, which is the unique element of* $C$ *such that*

$$\lim_{n\to\infty}(D_g(v_0, x_{n+1}) + \gamma_n D_g(v_0, x_n)) = \min_{v\in C}\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)).$$

*Proof* Since $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $g'$ is uniformly norm to norm continuous on bounded sets, we have that $\{g'(x_n)\}$ is bounded and

$$\|g'(x_n) - g'(x_{n+1})\|_* \to 0 \tag{2.16}$$

as $n \to \infty$. From the convexity of $g$, we deduce that

$$\langle g'(x_{n+1}), x_n - x_{n+1}\rangle \le g(x_n) - g(x_{n+1}) \le \langle g'(x_n), x_n - x_{n+1}\rangle. \tag{2.17}$$

Hence we have $g(x_n) - g(x_{n+1}) \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned} D_g(v, x_{n+1}) - D_g(v, x_n) &= g(x_n) - g(x_{n+1}) + \langle g'(x_n), v - x_n\rangle - \langle g'(x_{n+1}), v - x_{n+1}\rangle \\ &= g(x_n) - g(x_{n+1}) + \langle g'(x_n) - g'(x_{n+1}), v - x_n\rangle \\ &\quad + \langle g'(x_{n+1}), x_{n+1} - x_n\rangle, \end{aligned} \tag{2.18}$$

we have from (2.16–2.18) and the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n) - (1+\gamma_n)D_g(v, x_n)) = 0 \tag{2.19}$$

holds uniformly for any $v \in B \subseteq C$, where $B$ is an arbitrary bounded subset of $C$. Since $\{D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)\}$ is nonincreasing for any $v \in C$, it is convergent for any $v \in C$. By (2.19), we deduce

$$\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)) = \lim_{n\to\infty}(1+\gamma_n)D_g(v, x_n) \tag{2.20}$$

for any $v \in C$.

Let $f(v) = \lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n))$ for any $v \in C$. Then $f(v)$ is proper convex and nonnegative on $C$. By (2.20), we have

$$f(v) = \lim_{n\to\infty}(1+\gamma_n)D_g(v, x_n) \tag{2.21}$$

for any $v \in C$. Since $g'$ is uniformly norm to norm continuous on bounded sets, $g$ has bounded Gâteaux derivative $g'$ on bounded sets. From the proof of Proposition 2.1 in [11] and the boundedness of $\{\gamma_n\}$, we know that $f$ is continuous on $C$. Similarly, $f$ is coercive on $C$ (see Proposition 2.1 in [11]). Hence the function $f$ has a minimizer $v_0$ on the set $C$, i.e.,

$$f(v_0) = \min_{v\in C} f(v).$$

By (2.21), we get that $f(v_0) = \lim_{n\to\infty}(1+\gamma_n)D_g(v_0, x_n)$. Then, for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$(1+\gamma_n)D_g(v_0, x_n) < f(v_0) + \varepsilon \tag{2.22}$$

for all $n \ge N$. From the inequality (2.6) we get

$$D_g(v_0, \Pi_C^g x_n) + D_g(\Pi_C^g x_n, x_n) \le D_g(v_0, x_n) \tag{2.23}$$

for all $n \in \mathbb{N}$. It results by (2.22) and (2.23) that

$$(1+\gamma_n)D_g(\Pi_C^g x_n, x_n) < f(v_0) + \varepsilon \tag{2.24}$$

for all $n \ge N$.

From the total convexity on bounded sets of $g$ and the boundedness of $\{x_n\}$, similar to the proof of Proposition 2.1 in [11], we get that $\{\Pi_C^g x_n\}$ is bounded. For the subset $\{\Pi_C^g x_n\} \subseteq C$ and the above $\varepsilon$, by (2.19), there exists $N_1 > N$ such that

$$|D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n) - (1+\gamma_n) D_g(v, x_n)| < \varepsilon \tag{2.25}$$

for all $n \geq N_1$ and any $v \in \{\Pi_C^g x_n\}$. It follows from (2.24) and (2.25) that

$$\begin{aligned} D_g(\Pi_C^g x_n, x_{n+1}) + \gamma_n D_g(\Pi_C^g x_n, x_n) &< (1+\gamma_n) D_g(\Pi_C^g x_n, x_n) + \varepsilon \\ &< f(v_0) + 2\varepsilon \end{aligned} \tag{2.26}$$

for all $n \geq N_1$. By (2.26) and the monotonicity of $\{D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)\}$ for any $v \in C$, we have that

$$\begin{aligned} f(v_0) &\leq f(\Pi_C^g x_n) \leq D_g(\Pi_C^g x_n, x_{n+k}) + \gamma_n D_g(\Pi_C^g x_n, x_{n+k-1}) \\ &\leq \cdots \leq D_g(\Pi_C^g x_n, x_{n+1}) + \gamma_n D_g(\Pi_C^g x_n, x_n) \\ &< (1+\gamma_n) D_g(\Pi_C^g x_n, x_n) + \varepsilon \\ &< f(v_0) + 2\varepsilon \end{aligned} \tag{2.27}$$

for all $n \geq N_1$. Then $|(1+\gamma_n) D_g(\Pi_C^g x_n, x_n) - f(v_0)| < \varepsilon$ holds for $n \geq N_1$, which means that

$$\lim_{n\to\infty} (1+\gamma_n) D_g(\Pi_C^g x_n, x_n) = f(v_0). \tag{2.28}$$

It holds from (2.23) that

$$(1+\gamma_n) D_g(v_0, \Pi_C^g x_n) + (1+\gamma_n) D_g(\Pi_C^g x_n, x_n) \leq (1+\gamma_n) D_g(v_0, x_n) \tag{2.29}$$

for all $n \in \mathbb{N}$. It follows from (2.28) and (2.29) that $\lim_{n\to\infty} D_g(v_0, \Pi_C^g x_n) = 0$. The function $g$ is totally convex on bounded sets, therefore, it is sequentially consistent. It follows from the boundedness of $\{\Pi_C^g x_n\}$ that $\lim_{n\to\infty} \|v_0 - \Pi_C^g x_n\| = 0$. □

**Proposition 2.2** *Let $X$ be a reflexive Banach space and suppose that the function $g : X \to \mathbb{R}$ is lower semicontinuous and totally convex on bounded sets, and $g$ has bounded Gâteaux derivative $g'$ on bounded sets. Let $\{x_n\}$ be bounded and $C$ be a nonempty closed convex subset of $X$. Suppose that $\{D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)\}$ is nonincreasing for any $v \in C$, where $\{\gamma_n\}$ is a nonnegative sequence and $\lim_{n\to\infty} \gamma_n = 0$. Then $\{\Pi_C^g x_n\}$ converges strongly to $v_0 \in C$, which is the unique element of $C$ such that*

$$\lim_{n\to\infty} (D_g(v_0, x_{n+1}) + \gamma_n D_g(v_0, x_n)) = \min_{v\in C} \lim_{n\to\infty} (D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n)).$$

*Proof* By the inequality

$$g(\Pi_C^g x_m) - g(x_n) \leq \langle g'(\Pi_C^g x_m), \Pi_C^g x_m - x_n \rangle,$$

we deduce

$$\begin{aligned} D_g(\Pi_C^g x_m, x_n) &= g(\Pi_C^g x_m) - g(x_n) - \langle g'(x_n), \Pi_C^g x_m - x_n \rangle \\ &\leq \langle g'(\Pi_C^g x_m) - g'(x_n), \Pi_C^g x_m - x_n \rangle \\ &\leq \|g'(\Pi_C^g x_m) - g'(x_n)\| \cdot \|\Pi_C^g x_m - x_n\|. \end{aligned} \tag{2.30}$$

Since $\{x_n\}$ and $\{\Pi_C^g x_n\}$ are bounded, and $g$ has bounded Gâteaux derivative $g'$ on bounded sets, we obtain by (2.30) that $\{D_g(\Pi_C^g x_m, x_n)\}$ is uniformly bounded for all $m \in \mathbb{N}$.

Let $f(v) = \lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n))$ for any $v \in C$. Similar to the deduction of (2.30), $\{D_g(v, x_n)\}$ is bounded. It follows from $\lim_{n\to\infty}\gamma_n = 0$ that

$$f(v) = \lim_{n\to\infty} D_g(v, x_n) \tag{2.31}$$

for any $v \in C$. By the proof of Proposition 2.1 in [11], $f$ has a minimizer $v_0$ on the set $C$, i.e., $\lim_{n\to\infty} D_g(v_0, x_n) = f(v_0) = \min_{v\in C} f(v)$. Moreover, $\lim_{n\to\infty}\gamma_n D_g(\Pi_C^g x_m, x_n) = 0$ holds uniformly for any $m \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$D_g(v_0, x_n) < f(v_0) + \varepsilon \tag{2.32}$$

for all $n \geq N$, and

$$\gamma_n D_g(\Pi_C^g x_m, x_n) < \varepsilon \tag{2.33}$$

for all $m \in \mathbb{N}$ and $n \geq N$. Due to (2.23), (2.32) and (2.33), we have

$$\begin{aligned} f(v_0) &\leq f(\Pi_C^g x_{n+1}) \leq D_g(\Pi_C^g x_{n+1}, x_{n+k}) + \gamma_{n+k-1} D_g(\Pi_C^g x_{n+1}, x_{n+k-1}) \\ &\leq \cdots \leq D_g(\Pi_C^g x_{n+1}, x_{n+1}) + \gamma_n D_g(\Pi_C^g x_{n+1}, x_n) \\ &< f(v_0) + 2\varepsilon \end{aligned} \tag{2.34}$$

for all $n \geq N$. It results by (2.34) that, for all $n \geq N$,

$$|D_g(\Pi_C^g x_{n+1}, x_{n+1}) + \gamma_n D_g(\Pi_C^g x_{n+1}, x_n) - f(v_0)| < 2\varepsilon. \tag{2.35}$$

It follows from (2.33) that

$$|D_g(\Pi_C^g x_{n+1}, x_{n+1}) - f(v_0)| < 3\varepsilon \tag{2.36}$$

for all $n \geq N$. Similar proof to the last part of that of Proposition 2.1 yields $\lim_{n\to\infty}\|v_0 - \Pi_C^g x_n\| = 0$. □

Let $\{x_n\} \subset X$ be a sequence. Denote by $\overline{\{x_n\}}^{ws}$ the weak sequential closure of $\{x_n\}$, i.e., the set of all points $x$ for which there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $x$.

**Lemma 2.8** *Let $X$ be a Banach space and $g'$ be weakly sequentially continuous. Let $\{x_n\}$ be a sequence such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and its any subsequence has a weakly convergent subsequence. Suppose that $\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n))$ is finite for any $v \in \overline{\{x_n\}}^{ws}$, where $\{\gamma_n\}$ is a convergent sequence and $\lim_{n\to\infty}\gamma_n \neq -1$. Then $\{x_n\}$ is weakly convergent.*

*Proof* Clearly, $\overline{\{x_n\}}^{ws} \neq \emptyset$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ and $x_{n_j} \rightharpoonup w$ as $i, j \to \infty$. Then $z$ and $w$ belong to $\overline{\{x_n\}}^{ws}$. By given assumption, $\lim_{n\to\infty}(D_g(z, x_{n+1}) + \gamma_n D_g(z, x_n))$ and $\lim_{n\to\infty}(D_g(w, x_{n+1}) + \gamma_n D_g(w, x_n))$ exist.

Since $\|x_{n+1} - x_n\| \to 0$, we have $x_{n_j+1} \rightharpoonup w$ as $j \to \infty$. Since $g'$ is weakly sequentially continuous, we have that $\{g'(x_{n_j})\}$ and $\{g'(x_{n_j+1})\}$ converge weakly* to $g'(w)$ as $j \to \infty$. Then $\lim_{j\to\infty}\langle z - w, (1 + \gamma_{n_j})g'(w) - (g'(x_{n_j+1}) + \gamma_{n_j} g'(x_{n_j}))\rangle = 0$. From the equality

$$\begin{aligned} D_g(z, x_{n_j+1}) + \gamma_{n_j} D_g(z, x_{n_j}) &= (1 + \gamma_{n_j}) D_g(z, w) + (D_g(w, x_{n_j+1}) + \gamma_{n_j} D_g(w, x_{n_j})) \\ &\quad + \langle z - w, (1 + \gamma_{n_j})g'(w) - (g'(x_{n_j+1}) + \gamma_{n_j} g'(x_{n_j}))\rangle, \end{aligned} \tag{2.37}$$

we have

$$\begin{aligned}
&\lim_{n\to\infty}(D_g(z,x_{n+1})+\gamma_n D_g(z,x_n))\\
&\quad=\lim_{j\to\infty}(D_g(z,x_{n_j+1})+\gamma_{n_j}D_g(z,x_{n_j}))\\
&\quad=\lim_{j\to\infty}(1+\gamma_{n_j})D_g(z,w)+\lim_{j\to\infty}(D_g(w,x_{n_j+1})+\gamma_{n_j}D_g(w,x_{n_j}))\\
&\qquad+\lim_{j\to\infty}\langle z-w,(1+\gamma_{n_j})g'(w)-(g'(x_{n_j+1})+\gamma_{n_j}g'(x_{n_j}))\rangle\\
&\quad=(1+\lim_{n\to\infty}\gamma_n)D_g(z,w)+\lim_{n\to\infty}(D_g(w,x_{n+1})+\gamma_n D_g(w,x_n)).
\end{aligned}\tag{2.38}$$

Similarly, we also get

$$\begin{aligned}
\lim_{n\to\infty}(D_g(w,x_{n+1})+\gamma_n D_g(w,x_n))=(1+\lim_{n\to\infty}\gamma_n)D_g(w,z)+\lim_{n\to\infty}(D_g(z,x_{n+1})\\
+\gamma_n D_g(z,x_n)).
\end{aligned}\tag{2.39}$$

Adding (2.38) and (2.39) side by side we obtain that

$$\begin{aligned}
&\lim_{n\to\infty}(D_g(w,x_{n+1})+\gamma_n D_g(w,x_n))+\lim_{n\to\infty}(D_g(z,x_{n+1})+\gamma_n D_g(z,x_n))\\
&\quad=(1+\lim_{n\to\infty}\gamma_n)(D_g(w,z)+D_g(z,w))+\lim_{n\to\infty}(D_g(w,x_{n+1})+\gamma_n D_g(w,x_n))\\
&\qquad+\lim_{n\to\infty}(D_g(z,x_{n+1})+\gamma_n D_g(z,x_n)).
\end{aligned}$$

Since $\lim_{n\to\infty}\gamma_n\neq -1$, we conclude that $D_g(w,z)+D_g(z,w)=0$, which implies that $w=z$. □

**Lemma 2.9** *Suppose that $X$ is a Banach space and $g'$ is weakly sequentially continuous. Let $\{x_n\}$ be a sequence such that its any subsequence has a weakly convergent subsequence. Suppose that $\lim_{n\to\infty}(D_g(v,x_{n+1})+\gamma_n D_g(v,x_n))$ is finite for any $v\in\overline{\{x_n\}}^{ws}$, where $\{\gamma_n\}$ is a sequence and $\lim_{n\to\infty}\gamma_n=0$. Then $\{x_n\}$ is weakly convergent.*

*Proof* Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i}\rightharpoonup z$ and $x_{n_j}\rightharpoonup w$ as $i,j\to\infty$. Since $g'$ is weakly sequentially continuous, $\{g'(x_{n_j})\}$ converges weakly* to $g'(w)$ as $j\to\infty$. On the other hand, there exist a subsequence $\{x_{n_{j_l}-1}\}$ of $\{x_{n_j-1}\}$ and some $u\in X$ such that $x_{n_{j_l}-1}\rightharpoonup u$ as $l\to\infty$, and so $\{g'(x_{n_{j_l}-1})\}$ converges weakly* to $g'(u)$. It results by $\lim_{n\to\infty}\gamma_n=0$ that

$$\begin{aligned}
&\lim_{l\to\infty}\langle z-w,(1+\gamma_{n_{j_l}-1})g'(w)-(g'(x_{n_{j_l}})+\gamma_{n_{j_l}-1}g'(x_{n_{j_l}-1}))\rangle\\
&\quad=\lim_{l\to\infty}\langle z-w,g'(w)-g'(x_{n_{j_l}})\rangle+\lim_{l\to\infty}\gamma_{n_{j_l}-1}\langle z-w,g'(w)-g'(x_{n_{j_l}-1})\rangle=0.
\end{aligned}\tag{2.40}$$

Replacing $x_{n_j+1}$, $\gamma_{n_j}$ and $x_{n_j}$ by $x_{n_{j_l}}$, $\gamma_{n_{j_l}-1}$ and $x_{n_{j_l}-1}$ in (2.37), respectively, we get

$$\begin{aligned}
&\lim_{n\to\infty}(D_g(z,x_{n+1})+\gamma_n D_g(z,x_n))\\
&=\lim_{l\to\infty}(D_g(z,x_{n_{j_l}})+\gamma_{n_{j_l}-1}D_g(z,x_{n_{j_l}-1}))\\
&=\lim_{l\to\infty}(1+\gamma_{n_{j_l}-1})D_g(z,w)+\lim_{l\to\infty}(D_g(w,x_{n_{j_l}})+\gamma_{n_{j_l}-1}D_g(w,x_{n_{j_l}-1}))\\
&\quad+\lim_{l\to\infty}\langle z-w,(1+\gamma_{n_{j_l}-1})g'(w)-(g'(x_{n_{j_l}})+\gamma_{n_{j_l}-1}g'(x_{n_{j_l}-1}))\rangle\\
&=D_g(z,w)+\lim_{n\to\infty}(D_g(w,x_{n+1})+\gamma_n D_g(w,x_n)).
\end{aligned}\tag{2.41}$$

Similar to the last part of the deduction of Lemma 2.8, we conclude that $w = z$. □

**Corollary 2.1** *Let $X$ be a Banach space and suppose that $g$ has bounded Gâteaux derivative $g'$ on bounded sets and $g'$ is weakly sequentially continuous. Let $\{x_n\}$ be a bounded sequence such that its any subsequence has a weakly convergent subsequence. Suppose that $\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n))$ exists for any $v \in \overline{\{x_n\}}^{ws}$, where $\{\gamma_n\}$ is a sequence and $\lim_{n\to\infty} \gamma_n = 0$. Then $\{x_n\}$ is weakly convergent.*

*Proof* By the inequality

$$g(v) - g(x_n) \leq \langle g'(v), v - x_n \rangle,$$

and

$$\begin{aligned} D_g(v, x_n) &= g(v) - g(x_n) - \langle g'(x_n), v - x_n \rangle \\ &\leq \langle g'(v) - g'(x_n), v - x_n \rangle \\ &\leq \|g'(v) - g'(x_n)\| \cdot \|v - x_n\| \end{aligned} \tag{2.42}$$

for any $v \in X$, applying the boundedness of $g'$ on bounded sets, it is easy to verify that $\{D_g(v, x_n)\}$ is bounded. Consequently, $\lim_{n\to\infty}(D_g(v, x_{n+1}) + \gamma_n D_g(v, x_n))$ is finite for any $v \in \overline{\{x_n\}}^{ws}$. The conclusion follows from Lemma 2.9. □

In the sequel we consider the properties of the Bregman distance and the Bregman projection with respect to $\|\cdot\|^2$.

Let $X$ be a smooth Banach space. Consider now the following function

$$\phi(y, x) := \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2, \text{ for all } x, y \in X,$$

which is called the *Lyapunov function* in [1]. It is obvious from the definition of $\phi$ that the Bregman distance $D_g(y, x)$ is just $\phi(y, x)$ whenever $g = \|\cdot\|^2$, and $(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|x\| + \|y\|)^2$ for all $x, y \in X$.

It is well known that if a Banach space $X$ is uniformly convex, then $\|\cdot\|^2$ is totally convex on bounded sets.

The following Lemma shows that a uniformly convex Banach space is sequentially consistent.

**Lemma 2.10** ([4,10]) *Let $X$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Define a function $V: X \times X^* \to \mathbb{R}$ as follows:

$$V(x, x^*) := \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|_*^2$$

for all $x \in X$ and $x^* \in X^*$. Then it is obvious that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

In the following we assume that $X$ is a reflexive, strictly convex and smooth Banach space. Let $C$ be a closed convex subset of $X$. Considering $g = \|\cdot\|^2$, in this case, the Bregman projection $\Pi_C^g$ is exactly the operator

$$\Pi_C x := \arg\min_{y\in C} \phi(y, x),$$

which is said to be the *generalized projection* of $x$ on $C$ (see [1]).

In this case the inequality (2.6) is as follows:

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \ \forall\, y \in C, \tag{2.43}$$

which implies that the operator $\Pi_C$ gives the best approximation of $x \in X$ relative to the functional $\phi(y, x)$ (see [1]). Consequently, $\Pi_C$ is the conditionally non-expansive operator relative to the functional $\phi(y, x)$ in Banach spaces, i.e.,

$$\phi(y, \Pi_C x) \leq \phi(y, x), \ \forall\, y \in C.$$

The operator $\Pi_C : X \to C \subset X$ is identity on $C$, i.e., for every $x \in C$, $\Pi_C x = x$. In a Hilbert space $H$, $J$ is an identity operator, $\phi(y, x) = \|y - x\|^2$ and $\Pi_C$ coincides with the metric projection operator $P_C$.

Note that if $g = \frac{1}{2}\|\cdot\|^2$ (in this case $D_g(y, x) = \frac{1}{2}\phi(y, x)$), by the definition of the Bregman distance, the Bregman projection and the total convexity of $g$, we know that the Bregman projection is the same operator and $g$ is totally convex as in that case when $g = \|\cdot\|^2$.

The following result is of great importance (see [1]):

**Lemma 2.11** (Basic variational principle for the generalized projection) *Assume that $C$ is a closed convex subset of $X$. Then $\hat{x} = \Pi_C x$ is the generalized projection of $x$ on $C$ if and only if the inequality*

$$\langle J(x) - J(\hat{x}), y - \hat{x}\rangle \leq 0 \tag{2.44}$$

*holds for all $y \in C$.*

## 3 Convergence theorems

When $X$ is a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous and $g = \|\cdot\|^2$, we can easily see that in this case $D_g(x, y) = \phi(x, y)$ and Proposition 2.1, Proposition 2.2, Lemma 2.8 and Lemma 2.9 are all satisfied, in which $g$ verify the assumptions of the results in Sect. 2.

In addition, an operator $A : X \to X^*$ is called hemicontinuous if it is continuous along each line segment in $X$ with respect to the weak* topology of $X^*$. Let $S : X \rightrightarrows X^*$ be a set-valued mapping. $G(S)$ is always referred to be the graph of $S$.

**Lemma 3.1** *Let $X$ be a reflexive, strictly convex and smooth Banach space, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A : X \to X^*$ is an inverse-monotone operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as*

$$\begin{cases} x_0, x_1 \in C, \\ z_n = J^{-1}((1 - \beta_n)J(A_\alpha x_n) + \beta_n J(A_\alpha x_{n-1})), \\ x_{n+1} = \Pi_C z_n \end{cases} \tag{3.1}$$

*for all $n \in \mathbb{N}$, where $x_0, x_1$ are two arbitrary elements in $C$ and $\{\beta_n\} \subset [0, 1)$ is a nonincreasing sequence. Then the sequences $\{x_n\}$ and $\{A_\alpha x_n\}$ are bounded, and $\{\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})\}$ and $\{\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})\}$ are nonincreasing and have the same limit for any $v \in A_0$, where $A_0$ is the monotonicity pole of $A$ relative to constant $\alpha$.*

*Proof* Fix $v \in A_0$. Since $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$, applying Lemma 2.3 to $g = \frac{1}{2}\|\cdot\|^2$ and noticing $\{x_n\} \subset C$, we get that the inequality

$$\phi(v, A_\alpha x_n) + \phi(A_\alpha x_n, x_n) \le \phi(v, x_n) \tag{3.2}$$

holds for all $n \in \mathbb{N}$. By (2.43) and (3.2), we have

$$\begin{aligned} \phi(v, x_{n+1}) &= \phi(v, \Pi_C z_n) \\ &\le \phi(v, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\ &\le \phi(v, z_n) = V(v, J(z_n)) \\ &= V(v, (1-\beta_n)J(A_\alpha x_n) + \beta_n J(A_\alpha x_{n-1})) \\ &\le (1-\beta_n)V(v, J(A_\alpha x_n)) + \beta_n V(v, J(A_\alpha x_{n-1})) \\ &= (1-\beta_n)\phi(v, A_\alpha x_n) + \beta_n \phi(v, A_\alpha x_{n-1}) \\ &\le (1-\beta_n)\phi(v, x_n) + \beta_n \phi(v, x_{n-1}) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. Adding $\beta_n\phi(v, x_n)$ on both sides of (3.3) and noting that $\{\beta_n\}$ is nonincreasing, we obtain

$$\begin{aligned} \phi(v, x_{n+1}) + \beta_n\phi(v, x_n) &\le \phi(v, x_n) + \beta_n\phi(v, x_{n-1}) \\ &\le \phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1}). \end{aligned} \tag{3.4}$$

This implies that the nonnegative real number sequence $\{\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})\}$ is nonincreasing for any $v \in A_0$. So, it is convergent for any $v \in A_0$. By (3.2) and (3.3), we get, for any $n \in \mathbb{N}$,

$$\phi(v, A_\alpha x_{n+1}) \le \phi(v, x_{n+1}) \le (1-\beta_n)\phi(v, A_\alpha x_n) + \beta_n\phi(v, A_\alpha x_{n-1}). \tag{3.5}$$

Similarly,

$$\phi(v, A_\alpha x_{n+1}) + \beta_n\phi(v, A_\alpha x_n) \le \phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1}). \tag{3.6}$$

Since $(\|x_n\| - \|v\|)^2 \le \phi(v, x_n) \le \phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})$, from (3.4), we have that $\{x_n\}$ is bounded. From (3.6), $\{A_\alpha x_n\}$ is bounded.

It follows from (3.1), (3.3) and (3.2) that

$$\begin{aligned} &\phi(v, x_{n+2}) + \beta_{n+1}\phi(v, x_{n+1}) \le \phi(v, z_{n+1}) + \beta_{n+1}\phi(v, z_n) \\ &\quad = V(v, ((1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n))) + \beta_{n+1}V(v, ((1-\beta_n)J(A_\alpha x_n) \\ &\qquad + \beta_n J(A_\alpha x_{n-1}))) \le (1-\beta_{n+1})V(v, J(A_\alpha x_{n+1})) + \beta_{n+1}V(v, J(A_\alpha x_n)) \\ &\qquad + \beta_{n+1}(1-\beta_n)V(v, J(A_\alpha x_n)) + \beta_{n+1}\beta_n V(v, J(A_\alpha x_{n-1})) \\ &\quad \le (1-\beta_{n+1})(V(v, J(A_\alpha x_{n+1})) + \beta_{n+1}V(v, J(A_\alpha x_n))) + \beta_{n+1}(V(v, J(A_\alpha x_n)) \\ &\qquad + \beta_n V(v, J(A_\alpha x_{n-1}))) \le (1-\beta_{n+1})(\phi(v, A_\alpha x_{n+1}) \\ &\qquad + \beta_n\phi(v, A_\alpha x_n)) + \beta_{n+1}(\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})) \\ &\quad \le (1-\beta_{n+1})(\phi(v, x_{n+1}) + \beta_n\phi(v, x_n)) + \beta_{n+1}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})) \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$, where the second inequality in (3.7) is due to the convexity of $V(v, \cdot)$ and we also use the monotonicity of $\{\beta_n\}$. Hence we have by (3.7) that

$$\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})) = \lim_{n\to\infty}(\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})). \tag{3.8}$$

□

**Lemma 3.2** *Let $X$ be a reflexive, strictly convex and smooth Banach space, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A\colon X \to X^*$ is an operator such that $A_\alpha$ is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ for some $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1) and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $\emptyset \neq C_0 = A^{-1}0 \cap C$, then the sequences $\{x_n\}$ and $\{A_\alpha x_n\}$ are bounded, and $\{\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})\}$ and $\{\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})\}$ are nonincreasing for any $v \in C_0$, which have the same limit.*

*Proof* Take $v \in C_0$. Since $A_\alpha$ is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, noting that $v \in C_0 \subseteq A^{-1}0$ and $\{x_n\} \subset C$, we have that

$$\phi(v, A_\alpha x_n) = \phi(A_\alpha v, A_\alpha x_n) \le \phi(v, x_n) \tag{3.9}$$

for any $n \in \mathbb{N}$. By (3.1), (2.43) and (3.9), we get that

$$\begin{aligned}\phi(v, x_{n+1}) &\le \phi(v, z_n)\\ &\le (1-\beta_n)\phi(v, A_\alpha x_n) + \beta_n\phi(v, A_\alpha x_{n-1})\\ &\le (1-\beta_n)\phi(v, x_n) + \beta_n\phi(v, x_{n-1})\end{aligned} \tag{3.10}$$

for any $n \in \mathbb{N}$. By using similar arguments as those in the proof of Lemma 3.1, we deduce by (3.10) and the monotonicity of $\{\beta_n\}$ that $\{\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})\}$ and $\{\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})\}$ are nonincreasing for $v \in C_0$, and so $\{x_n\}$ and $\{A_\alpha x_n\}$ are bounded.

The same inequality as (3.7) follows from (3.1), (3.10) and (3.9), in which the last inequality is due to (3.9), and so the equality (3.8) follows. □

**Theorem 3.1** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A\colon X \to X^*$ is a hemicontinuous, monotone and inverse-monotone operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $A^{-1}0 \cap C = A_0$, then the sequence $\{x_n\}$ converges weakly to $u$, which is the unique element of $A_0$ such that*

$$\lim_{n\to\infty}(\phi(u, x_n) + \beta_{n-1}\phi(u, x_{n-1})) = \min_{v\in A_0}\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})).$$

*Proof* We take $v \in A_0$. From (3.8) in Lemma 3.1 and the inequality

$$\begin{aligned}\phi(A_\alpha x_n, x_n) &\le \phi(v, x_n) - \phi(v, A_\alpha x_n)\\ &\le (\phi(v, x_n) - \phi(v, A_\alpha x_n)) + \beta_{n-1}(\phi(v, x_{n-1}) - \phi(v, A_\alpha x_{n-1}))\\ &= (\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})) - (\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})),\end{aligned} \tag{3.11}$$

it follows that

$$\lim_{n\to\infty}\phi(A_\alpha x_n, x_n) = 0. \tag{3.12}$$

Since $\{x_n\}$ is bounded, we get by Lemma 2.10 that

$$\lim_{n\to\infty}\|A_\alpha x_n - x_n\| = 0. \tag{3.13}$$

By the uniform smoothness of $X$, it follows from (3.13) that

$$\begin{aligned}\lim_{n\to\infty}\alpha\|Ax_n\|_* &= \lim_{n\to\infty}\|(Jx_n-\alpha Ax_n)-J(x_n)\|_*\\ &= \lim_{n\to\infty}\|J(J^{-1}(Jx_n-\alpha Ax_n))-J(x_n)\|_*\\ &= \lim_{n\to\infty}\|J(A_\alpha x_n)-J(x_n)\|_*=0. \qquad (3.14)\end{aligned}$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges weakly to some $u\in C$.

We next prove $u\in A^{-1}0$. Let

$$Sv=\begin{cases}Av+N_Cv, & \text{if } v\in C,\\ \emptyset, & \text{if } v\notin C.\end{cases} \qquad (3.15)$$

Since $A$ is hemicontinuous and monotone (see [14]), $S$ is maximal monotone and $v\in S^{-1}0$ if and only if $v\in VI(C,A)$. Since $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, by Lemma 2.5, $v\in S^{-1}0$ if and only if $v\in A^{-1}0\cap C$. Let $(v,v^*)\in G(S)$. Then, we have

$$v^*\in Sv=Av+N_Cv,$$

and hence $v^*-Av\in N_Cv$. Thus, we have

$$\langle v-w, v^*-Av\rangle\geq 0 \qquad (3.16)$$

for all $w\in C$. From $\{x_n\}\subset C$ and (3.16), we have

$$\begin{aligned}\langle v-x_{n_i}, v^*\rangle &\geq \langle v-x_{n_i}, Av\rangle\\ &= \langle v-x_{n_i}, Av-Ax_{n_i}\rangle+\langle v-x_{n_i}, Ax_{n_i}\rangle\\ &\geq \langle v-x_{n_i}, Ax_{n_i}\rangle \qquad (3.17)\end{aligned}$$

for all $i\in\mathbb{N}$. Therefore, letting $i\to\infty$, we obtain by (3.14) that $\langle v-u, v^*\rangle\geq 0$. Since $S$ is maximal monotone, we have $u\in S^{-1}0$ and hence $u\in A^{-1}0\cap C=A_0$. This implies that $\overline{\{x_n\}}^{ws}$ is included in $A_0$. So, we get that $\{\phi(v,x_{n+1})+\beta_n\phi(v,x_n)\}$ is convergent for any $v\in\overline{\{x_n\}}^{ws}$.

We distinguish the following two possible situations: Either (i) $\lim_{n\to\infty}\beta_n>0$ or (ii) $\lim_{n\to\infty}\beta_n=0$.

Case (i): Due to (3.7) and the fact that both sides of (3.7) have the same limit, both sides of the second inequality in (3.7) have also the same limit. Then the right side of the second inequality in (3.7) minus the left side tends to 0, that is

$$\begin{aligned}&((1-\beta_{n+1})V(v,J(A_\alpha x_{n+1}))+\beta_{n+1}V(v,J(A_\alpha x_n))-V(v,((1-\beta_{n+1})J(A_\alpha x_{n+1})+\\ &\beta_{n+1}J(A_\alpha x_n))))+(\beta_{n+1}(1-\beta_n)V(v,J(A_\alpha x_n))+\beta_{n+1}\beta_nV(v,J(A_\alpha x_{n-1}))\\ &\quad-\beta_{n+1}V(v,((1-\beta_n)J(A_\alpha x_n)+\beta_nJ(A_\alpha x_{n-1}))))\to 0 \qquad (3.18)\end{aligned}$$

In fact, the right side of the second inequality in (3.7) minus the left side is just (3.18).

Moreover, in lines −1−−9 of page 14 and line 1 of page 15, we have changed the following proof

"Noting that the two terms in (3.18) are both nonnegative, we deduce by (3.18) that

$$\begin{aligned}
&(1-\beta_{n+1})\|J(A_\alpha x_{n+1})\|_*^2 + \beta_{n+1}\|J(A_\alpha x_n)\|_*^2 - \|(1-\beta_{n+1})J(A_\alpha x_{n+1})\\
&\quad +\beta_{n+1}J(A_\alpha x_n)\|_*^2\\
&\quad = (1-\beta_{n+1})(\|v\|^2 - 2\langle v, J(A_\alpha x_{n+1})\rangle + \|J(A_\alpha x_{n+1})\|_*^2)\\
&\qquad + \beta_{n+1}(\|v\|^2 - 2\langle v, J(A_\alpha x_n)\rangle + \|J(A_\alpha x_n)\|_*^2)\\
&\qquad - (\|v\|^2 - 2\langle v, ((1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n))\rangle\\
&\qquad + \|(1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n)\|_*^2)\\
&\quad = ((1-\beta_{n+1})V(v, J(A_\alpha x_{n+1})) + \beta_{n+1}V(v, J(A_\alpha x_n))\\
&\qquad - V(v, ((1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n)))) \to 0,
\end{aligned}$$

which is the first term in (3.18)" as:

" Noting that the two terms in (3.18) are both nonnegative 'by the convexity of $V(v, \cdot)$', we deduce by (3.18) that

$$\begin{aligned}
0 &\leftarrow (1-\beta_{n+1})V(v, J(A_\alpha x_{n+1})) + \beta_{n+1}V(v, J(A_\alpha x_n))\\
&\quad - V(v, ((1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n)))\\
&= (1-\beta_{n+1})(\|v\|^2 - 2\langle v, J(A_\alpha x_{n+1})\rangle + \|J(A_\alpha x_{n+1})\|_*^2)\\
&\quad + \beta_{n+1}(\|v\|^2 - 2\langle v, J(A_\alpha x_n)\rangle + \|J(A_\alpha x_n)\|_*^2)\\
&\quad - (\|v\|^2 - 2\langle v, ((1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n))\rangle\\
&\quad + \|(1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n)\|_*^2)\\
&= (1-\beta_{n+1})\|J(A_\alpha x_{n+1})\|_*^2 + \beta_{n+1}\|J(A_\alpha x_n)\|_*^2 - \|(1-\beta_{n+1})J(A_\alpha x_{n+1})\\
&\quad + \beta_{n+1}J(A_\alpha x_n)\|_*^2,
\end{aligned}$$

which is the first term in (3.18)". I.e., the sum of the two nonnegative terms in (3.18) tends to 0 implies each term converges to 0. Since $X$ is uniformly smooth, $X^*$ is uniformly convex. Since $\{J(A_\alpha x_n)\}$ is bounded, by Lemma 2.1, there exists a strictly increasing, continuous and convex function $k^*\colon [0,\infty) \to [0,\infty)$ such that $k^*(0) = 0$ and

$$\begin{aligned}
&\|(1-\beta_{n+1})J(A_\alpha x_{n+1}) + \beta_{n+1}J(A_\alpha x_n)\|_*^2\\
&\quad \le (1-\beta_{n+1})\|J(A_\alpha x_{n+1})\|_*^2 + \beta_{n+1}\|J(A_\alpha x_n)\|_*^2\\
&\qquad - \beta_{n+1}(1-\beta_{n+1})k^*(\|J(A_\alpha x_{n+1}) - J(A_\alpha x_n)\|_*)
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $k^*$ is strictly increasing, it follows from $k^*(0) = 0$ and $\lim_{n\to\infty} \beta_n > 0$ that $\|J(A_\alpha x_{n+1}) - J(A_\alpha x_n)\|_* \to 0$ as $n \to \infty$. Since $X$ is uniformly convex, $X^*$ is uniformly smooth. So, we have $\|A_\alpha x_{n+1} - A_\alpha x_n\| \to 0$ as $n \to \infty$. Hence we deduce by (3.13) that

$$\|x_{n+1} - x_n\| \le \|A_\alpha x_{n+1} - x_{n+1}\| + \|A_\alpha x_{n+1} - A_\alpha x_n\| + \|A_\alpha x_n - x_n\| \to 0. \tag{3.19}$$

It follows from Lemma 2.8 that $x_n \rightharpoonup u$.

By the definition of $A_0$, we know that it is a closed convex subset of $C$. Put $u_n = \Pi_{A_0} x_n$. From (2.44) and $u \in A_0$, We have

$$\langle u - u_n, J(u_n) - J(x_n)\rangle \ge 0. \tag{3.20}$$

By Proposition 2.1, $\{u_n\}$ converges strongly to some $u_0$, which is the unique element of $A_0$ such that

$$\lim_{n\to\infty}(\phi(u_0, x_{n+1}) + \beta_n\phi(u_0, x_n)) = \min_{v\in A_0}\lim_{n\to\infty}(\phi(v, x_{n+1}) + \beta_n\phi(v, x_n)). \quad (3.21)$$

Since $J$ is uniformly norm to norm continuous on bounded sets and weakly sequentially continuous, by (3.20), we conclude

$$\langle u - u_0, J(u_0) - J(u)\rangle \geq 0.$$

It follows from the strict monotonicity of $J$ that $u = u_0$.

Case (ii): Noting that $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}(\phi(v, x_{n+1}) + \beta_n\phi(v, x_n))$ is finite for any $v \in \overline{\{x_n\}}^{ws}$, we obtain by Lemma 2.9 that $x_n \rightharpoonup u$.

In a similar way as in the last part of the proof of case (i), by Proposition 2.2, $\{u_n\}$ converges strongly to $u$, where $u_n = \Pi_{A_0}x_n$ and $u$ is the unique element of $A_0$ such that (3.21) holds. So, the conclusion follows. □

**Corollary 3.1** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A: X \to X^*$ is a hemicontinuous, monotone and strongly inverse-monotone operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $\emptyset \neq C_0 = A^{-1}0 \cap C$, then the sequence $\{x_n\}$ converges weakly to $u$, which is the unique element of $C_0$ such that*

$$\lim_{n\to\infty}(\phi(u, x_n) + \beta_{n-1}\phi(u, x_{n-1})) = \min_{v\in C_0}\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})).$$

*Proof* Since $A^{-1}0 \cap C \neq \emptyset$ and $A$ is strongly inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, by Lemma 2.6, $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ and $A^{-1}0 \cap C = A_0$. By Theorem 3.1, the conclusion follows. □

**Theorem 3.2** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A: X \to X^*$ is an operator such that $A_\alpha$ is strongly nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ for some $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $\emptyset \neq C_0 = A^{-1}0 \cap C$, then the sequence $\{x_n\}$ converges weakly to $u \in C_0$. In addition, if $C_0$ is a closed convex subset of $C$, then $u$ is the unique element of $C_0$ such that*

$$\lim_{n\to\infty}(\phi(u, x_n) + \beta_{n-1}\phi(u, x_{n-1})) = \min_{v\in C_0}\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})). \quad (3.22)$$

*Proof* Due to Lemma 3.2, $\{x_n\}$ is bounded, and so $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges weakly to some $u \in C$. Next we prove $u \in A^{-1}0$. Since $A_\alpha$ is strongly nonexpansive on $C$, $v \in C_0 \subseteq A^{-1}0$ and $\{x_n\} \subset C$, there exists some $\lambda > \alpha$ such that

$$\begin{aligned}\phi(v, A_\alpha x_n) &= \phi(A_\alpha v, A_\alpha x_n)\\ &\leq \phi(v, x_n) + \alpha(\alpha - \lambda)\phi(J^{-1}(Av), J^{-1}(Ax_n))\\ &\leq \phi(v, x_n) \end{aligned} \quad (3.23)$$

for any $n \in \mathbb{N}$. Then, by (3.23) and Lemma 3.2, we have from the nonnegativity of $\{\beta_n\}$ that

$$\begin{aligned} 0 &\le \alpha(\lambda-\alpha)\phi(J^{-1}(Av), J^{-1}(Ax_n)) \\ &\le \phi(v, x_n) - \phi(v, A_\alpha x_n) \\ &\le \phi(v, x_n) - \phi(v, A_\alpha x_n) + \beta_{n-1}(\phi(v, x_{n-1}) - \phi(v, A_\alpha x_{n-1})) \\ &= (\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})) - (\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})) \to 0 \end{aligned} \tag{3.24}$$

as $n \to \infty$, noting that $v \in A^{-1}0$, and hence we deduce by (3.24) that

$$\|Ax_n\|_*^2 = \phi(J^{-1}(Av), J^{-1}(Ax_n)) \to 0. \tag{3.25}$$

By (3.25) and Lemma 3.2, we also have

$$\begin{aligned} 0 &\le \phi(A_\alpha x_n, x_n) \\ &= \phi(v, x_n) - \phi(v, A_\alpha x_n) - 2\langle v - A_\alpha x_n, J(A_\alpha x_n) - J(x_n)\rangle \\ &\le \phi(v, x_n) - \phi(v, A_\alpha x_n) + 2\|v - A_\alpha x_n\| \cdot \|J(A_\alpha x_n) - J(x_n)\|_* \\ &\le \phi(v, x_n) - \phi(v, A_\alpha x_n) + M\|J(A_\alpha x_n) - J(x_n)\|_* \\ &= \phi(v, x_n) - \phi(v, A_\alpha x_n) + M\alpha\|Ax_n\|_* \\ &\le \phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1}) - (\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})) \\ &\quad + M\alpha\|Ax_n\|_* \to 0 \end{aligned} \tag{3.26}$$

as $n \to \infty$, where $M = \sup_{n\ge1} 2\|v - A_\alpha x_n\|$ and the first equality is due to the definition of $\phi$. This yields $\phi(A_\alpha x_n, x_n) \to 0$. Since $\{x_{n_i}\}$ converges weakly to $u$, it follows from Lemma 2.7 that $u$ is a fixed point of $A_\alpha$. Hence $u \in A^{-1}0$ and so $u \in C_0$. This implies that $\overline{\{x_n\}}^{ws}$ is included in $C_0$. Thus we have that $\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1}))$ exists for any $v \in \overline{\{x_n\}}^{ws}$.

Suppose that $\lim_{n\to\infty}\beta_n > 0$. By the same arguments as those of Theorem 3.1, it follows from the same inequality as (3.7) and the uniform convexity of $X^*$ that $\|J(A_\alpha x_{n+1}) - J(A_\alpha x_n)\|_* \to 0$ as $n \to \infty$. This shows together with (3.25) that

$$\begin{aligned} \|J(x_{n+1}) - J(x_n)\|_* &= \|(J(x_{n+1}) - \alpha Ax_{n+1}) + \alpha Ax_{n+1} - (J(x_n) - \alpha Ax_n) - \alpha Ax_n\|_* \\ &\le \|J(A_\alpha x_{n+1}) - J(A_\alpha x_n)\|_* + \alpha\|Ax_{n+1}\|_* + \alpha\|Ax_n\|_* \to 0. \end{aligned} \tag{3.27}$$

Then it holds from the uniform smoothness of $X^*$ that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.28}$$

It follows from Lemma 2.8 that $x_n \rightharpoonup u$.

Suppose that $C_0$ is a closed convex subset of $C$. From the last part of the proof of case (i) of Theorem 3.1, by Proposition 2.1, $\{u_n\}$ converges strongly to some $u$, where $u_n = \Pi_{C_0}x_n$ and $u$ is the unique element of $C_0$ such that

$$\lim_{n\to\infty}(\phi(u, x_n) + \beta_{n-1}\phi(u, x_{n-1})) = \min_{v\in C_0}\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})). \tag{3.29}$$

When $\lim_{n\to\infty}\beta_n = 0$, noting that $\lim_{n\to\infty}(\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1}))$ is finite for any $v \in \overline{\{x_n\}}^{ws}$, it follows from Lemma 2.9 that $x_n \rightharpoonup u$.

Suppose that $C_0$ is a closed convex subset of $C$. By Proposition 2.2, $\{u_n\}$ converges strongly to $u$, where $u_n = \Pi_{C_0} x_n$ and $u$ is the unique element of $C_0$ such that (3.29) holds. So, the conclusion follows. □

**Corollary 3.2** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A\colon X \to X^*$ is a hemicontinuous and monotone operator such that $A_\alpha$ is strongly nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ for some $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $\emptyset \neq C_0 = A^{-1}0 \cap C$, then the sequence $\{x_n\}$ converges weakly to $u$, which is the unique element of $C_0$ such that (3.22) holds.*

*Proof* From Theorem 3.2, it is sufficient to show that $C_0$ is a closed convex subset of $C$. Let

$$Sv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Since $A$ is hemicontinuous and monotone, $S$ is maximal monotone and $v \in S^{-1}0$ if and only if $v \in VI(C, A)$. Since $A_\alpha$ is a strongly nonexpansive operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, it is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$. Noting that $A^{-1}0 \cap C \neq \emptyset$, by Lemma 2.4, we get that $v \in S^{-1}0$ if and only if $v \in A^{-1}0 \cap C$. The maximal monotonicity of $S$ guarantees that $C_0 = A^{-1}0 \cap C = S^{-1}0$ is a closed convex subset of $C$. □

**Theorem 3.3** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A\colon X \to X^*$ is an inverse strongly-monotone operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. Then the sequence $\{x_n\}$ converges weakly to $u$. In addition, if $C_0 = A^{-1}0 \cap C$ is a closed convex subset of $C$, especially $C_0 = A_0$, then $u$ is the unique element of $C_0$ such that*

$$\lim_{n\to\infty} (\phi(u, x_n) + \beta_{n-1}\phi(u, x_{n-1})) = \min_{v\in C_0} \lim_{n\to\infty} (\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})), \quad (3.30)$$

*where $A_0$ is monotonicity pole of $A$.*

*Proof* Since $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$, by Theorem 5.5 in [5], we have $C_0 \neq \emptyset$. Take $v \in C_0$. Since $A_\alpha$ is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, by Lemma 3.2, we have that $\lim_{n\to\infty} (\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1}))$ and $\lim_{n\to\infty} (\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1}))$ exist for any $v \in C_0$, and

$$\lim_{n\to\infty} (\phi(v, x_n) + \beta_{n-1}\phi(v, x_{n-1})) = \lim_{n\to\infty} (\phi(v, A_\alpha x_n) + \beta_{n-1}\phi(v, A_\alpha x_{n-1})). \quad (3.31)$$

Fix $v \in A_0 \subseteq C_0$. Since $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ and $\{x_n\} \subset C$, we have that (3.11) holds true. Due to (3.11) and (3.31), we get that

$$\lim_{n\to\infty} \phi(A_\alpha x_n, x_n) = 0. \quad (3.32)$$

This means by Lemma 2.10 that

$$\lim_{n\to\infty} \|A_\alpha x_n - x_n\| = 0. \quad (3.33)$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges weakly to some $u \in C$. Since $A_\alpha$ is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, from (3.32), Lemma 2.7 yields that

$u$ is a fixed point of $A_\alpha$. Hence $u \in A^{-1}0$, and so $u \in C_0$. This shows that $\overline{\{x_n\}}^{ws}$ is included in $C_0$. Hence we have that $\lim_{n\to\infty}(\phi(v,x_n)+\beta_{n-1}\phi(v,x_{n-1}))$ exists for any $v \in \overline{\{x_n\}}^{ws}$.

Suppose that $\lim_{n\to\infty}\beta_n > 0$. Since $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, by the same proof as that of Theorem 3.1, it follows that $\lim_{n\to\infty}\|A_\alpha x_{n+1} - A_\alpha x_n\| = 0$. Similar to (3.19), we have together with (3.33) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.34}$$

It follows from Lemma 2.8 that $x_n \rightharpoonup u$.

If $C_0$ is a closed convex subset of $C$, similar to the proof of case (i) of Theorem 3.1, the conclusion follows from Proposition 2.1. Especially, when $C_0 = A_0$, by the closedness and convexity of $A_0$, we conclude that $C_0$ is a closed convex subset of $C$. So the conclusion holds.

When $\lim_{n\to\infty}\beta_n = 0$, noting that $\lim_{n\to\infty}(\phi(v,x_n)+\beta_{n-1}\phi(v,x_{n-1}))$ is finite for any $v \in \overline{\{x_n\}}^{ws}$, it follows from Lemma 2.9 that $x_n \rightharpoonup u$.

If $C_0$ is a closed convex subset of $C$, especially, $C_0 = A_0$, the conclusion follows from Proposition 2.2. □

**Corollary 3.3** *Let $X$ be a uniformly smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous, and $C$ be a nonempty closed convex subset of $X$. Suppose that $A\colon X \to X^*$ is a strongly inverse-monotone operator on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ with constant $\alpha > 0$ such that $A_\alpha$ is nonexpansive on $C$ relative to $\frac{1}{2}\|\cdot\|^2$. Let $\{x_n\}$ be a sequence defined as (3.1), and $\{\beta_n\}$ be chosen according to Lemma 3.1. If $\emptyset \neq C_0 = A^{-1}0 \cap C$, then the sequence $\{x_n\}$ converges weakly to $u$, which is the unique element of $C_0$ such that (3.30) holds.*

*Proof* Since $A^{-1}0 \cap C \neq \emptyset$ and $A$ is strongly inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$, by Lemma 2.6, $A$ is inverse-monotone on $C$ relative to $\frac{1}{2}\|\cdot\|^2$ and $A^{-1}0 \cap C = A_0$. Thus, $\emptyset \neq C_0 = A^{-1}0 \cap C = A_0$. By Theorem 3.3, the conclusion follows. □

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